Applications of Lagrangian duality in Boolean function analysis

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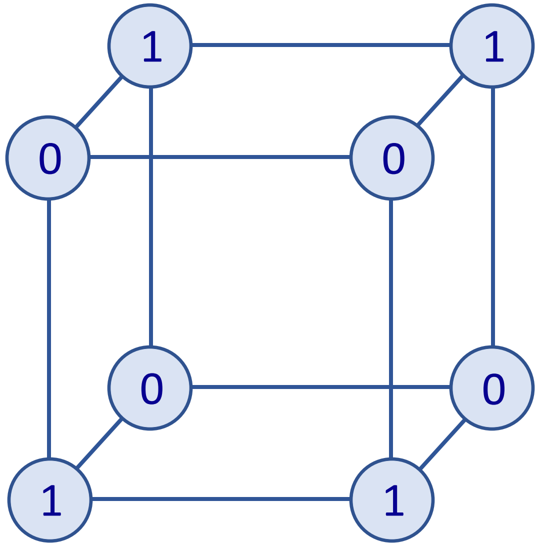

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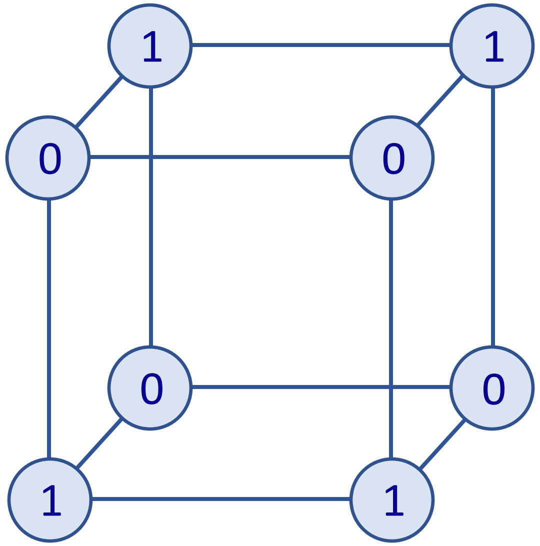

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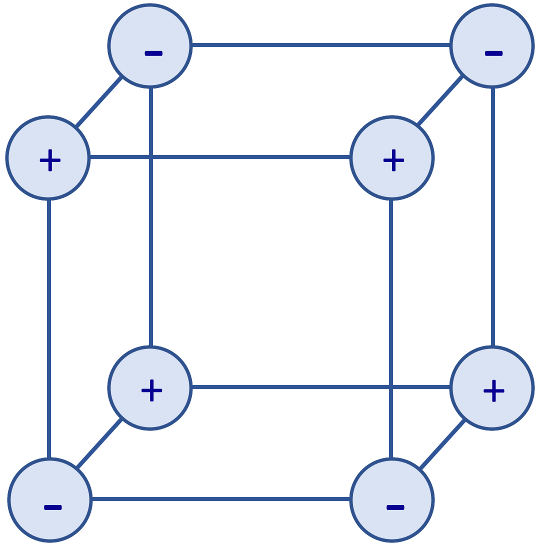

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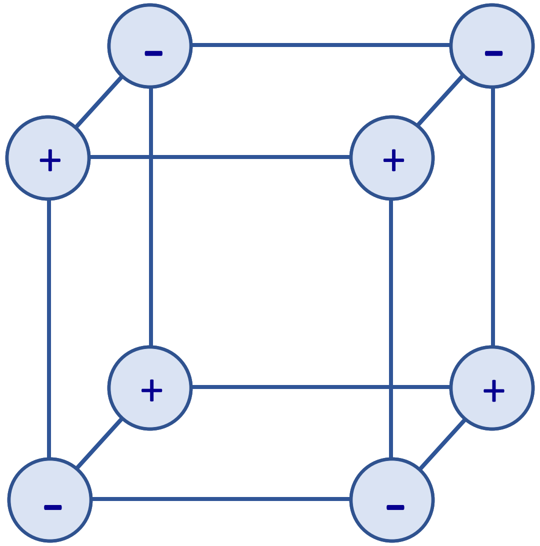

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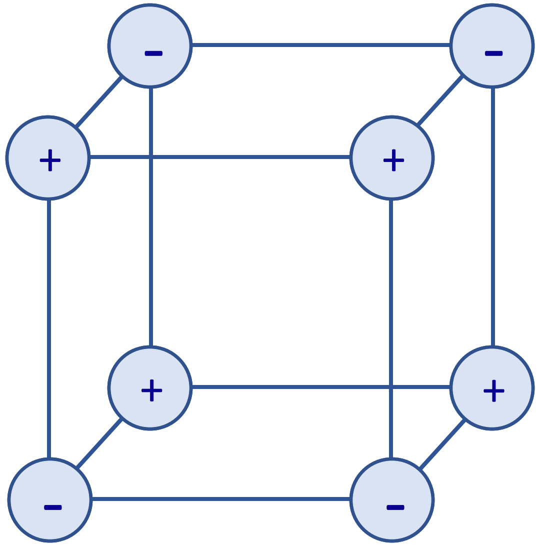

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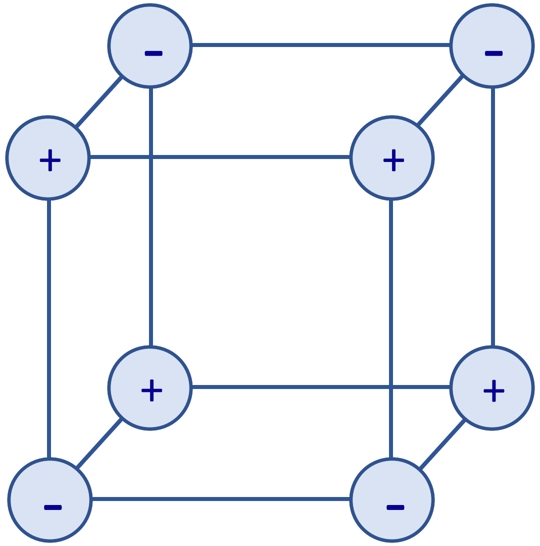

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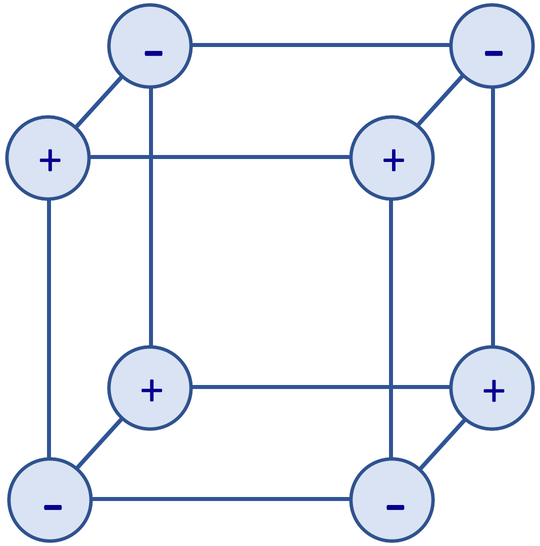

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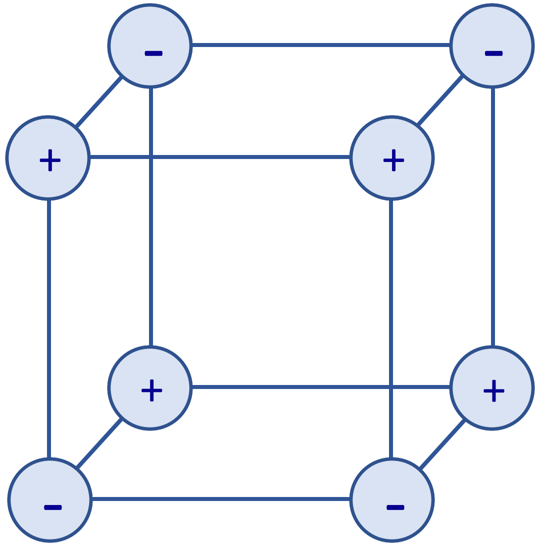

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$D(f) \geq \deg(f)$
$Q_0(f) \geq \deg(f)/2$

Inner product & norms

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Inner product: for $f, g: \{-1, 1\}^n \to \mathbb{R}$, define

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{\boldsymbol{x} \sim \{-1,1\}^n} [f(\boldsymbol{x}) g(\boldsymbol{x})] = 2^{-n} \sum_{x \in \{-1,1\}^n} f(x) g(x)$$

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***p*-norm:** for $f: \{-1,1\}^n \to \mathbb{R}$, $p \in [1, \infty]$, define

$$\|f\|_p = \mathop{\mathbb{E}}_{\boldsymbol{x} \sim \{-1,1\}^n}[|f(\boldsymbol{x})|^p]^{1/p}$$

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$$\begin{aligned}
\widehat{f * g}(S) &= \mathop{\mathbb{E}}_{\boldsymbol{x}\sim\{-1,1\}^n}[(f * g)(\boldsymbol{x})\chi_S(\boldsymbol{x})] \\
&= \mathop{\mathbb{E}}_{\boldsymbol{x},\boldsymbol{y}\sim\{-1,1\}^n}[f(\boldsymbol{y})g(\boldsymbol{x}\boldsymbol{y})\chi_S(\boldsymbol{x})] \\
&= \mathop{\mathbb{E}}_{\boldsymbol{y},\boldsymbol{z}\sim\{-1,1\}^n}[f(\boldsymbol{y})g(\boldsymbol{z})\chi_S(\boldsymbol{y}\boldsymbol{z})] \\
&= \mathop{\mathbb{E}}_{\boldsymbol{y},\boldsymbol{z}\sim\{-1,1\}^n}[f(\boldsymbol{y})\chi_S(\boldsymbol{y})g(\boldsymbol{z})\chi_S(\boldsymbol{z})] \\
&= \hat{f}(S)\hat{g}(S).
\end{aligned}$$

Two equivalent propositions

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P1: For ***all*** $f\colon \{-1,1\}^n \to \mathbb{R}$, $\deg(f) \leq d$, $\|h * f\|_{p_1} \leq c\|f\|_{p_2}$.

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P2: For all $Z: \{-1,1\}^n \to \mathbb{R}$ with $\|Z\|_{q_1} \leq 1$, there ***exists*** $g_Z: \{-1,1\}^n \to \mathbb{R}$ such that

1. $\|g_Z\|_{q_2} \leq c$, and
2. $\widehat{g_Z}(S) = \hat{Z}(S) \cdot \hat{h}(S)$, for all $S \subseteq [n]$, $|S| \leq d$.

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Theorem (Nisan-Szegedy, 1994): For all $f: \{-1, 1\}^n \to \{-1, 1\}$,
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An equivalent form: For all $f: \{-1,1\}^n \to \mathbb{R}$, $\deg(f) \leq d$,

$$\left\| \sum_{S \subseteq [n]} |S| \hat{f}(S) \chi_S \right\|_\infty \leq O(d^2) \|f\|_\infty$$

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$$\|h * f\|_\infty \le O(d^2)\|f\|_\infty$$

$$h \leftarrow \sum_{S \subseteq [n]} |S| \chi_S$$

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Dual: For all $Z: \{-1,1\}^n \to \mathbb{R}$ with $\|Z\|_1 \leq 1$, there exists $g_Z: \{-1,1\}^n \to \mathbb{R}$ such that

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We only need: There exists $p: \{-1,1\}^n \to \mathbb{R}$ such that

1. $\|p\|_1 \leq d^2$, and
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$$g_Z \leftarrow Z * p$$

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$$\boldsymbol{w} = \left(\frac{2d^2+1}{6}, -\csc^2\left(\frac{\pi}{2d}\right), \cdots, (-1)^i \csc^2\left(\frac{i\pi}{2d}\right), \cdots, (-1)^{d-1}\csc^2\left(\frac{(d-1)\pi}{2d}\right), \frac{(-1)^d}{2}\right)^{\mathsf{T}}$$

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Theorem: For all $f: \{-1, 1\}^n \to \mathbb{R}$,

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$$\|h * f\|_\infty \leq O(\log d) \|f\|_\infty$$

Find $\boldsymbol{p}$

We need to find $\{(w_i, \alpha_i)\}_{i\in\Gamma}, |\alpha_i| \le 1$ such that

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Fix $\{\alpha_i\}_{i\in\Gamma}$:

$$\begin{aligned} \min \quad & \|\boldsymbol{w}\|_1, \\ \text{s.t.} \quad & A\boldsymbol{w} = \boldsymbol{b}, \end{aligned}$$

where $A = \left(\alpha_j^i\right)_{1\le i\le d, j\in\Gamma}$, $\boldsymbol{b} = (1, 1/2, \dots, 1/d)^\mathsf{T}$.

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$$\begin{array}{ll} \max & \boldsymbol{b}^\mathsf{T}\boldsymbol{z} \\ \text{s.t.} & \|A^\mathsf{T}\boldsymbol{z}\|_\infty \leq 1 \end{array}$$

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$$\leq \sum_{m=1}^{K} \frac{1}{m} + \sum_{k=1}^{d} \left(\frac{2d^2}{K}\right)^k = O(\log d)$$

$K \leftarrow 4d^2$

Thank you!